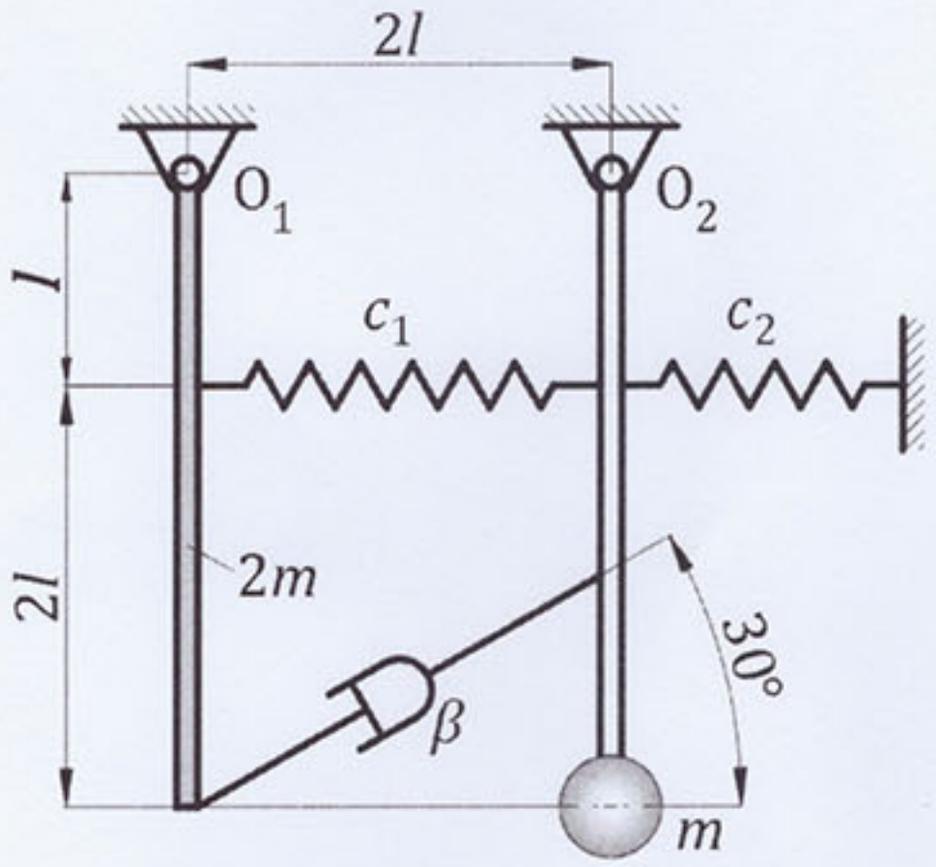
Универзитет Црне Горе
Машински факултет Подгорица 19.06.2020.

ДРУГИ КОЛОКВИЈУМ ИЗ ОСЦИЛАЦИЈА У МАШИНСТВУ

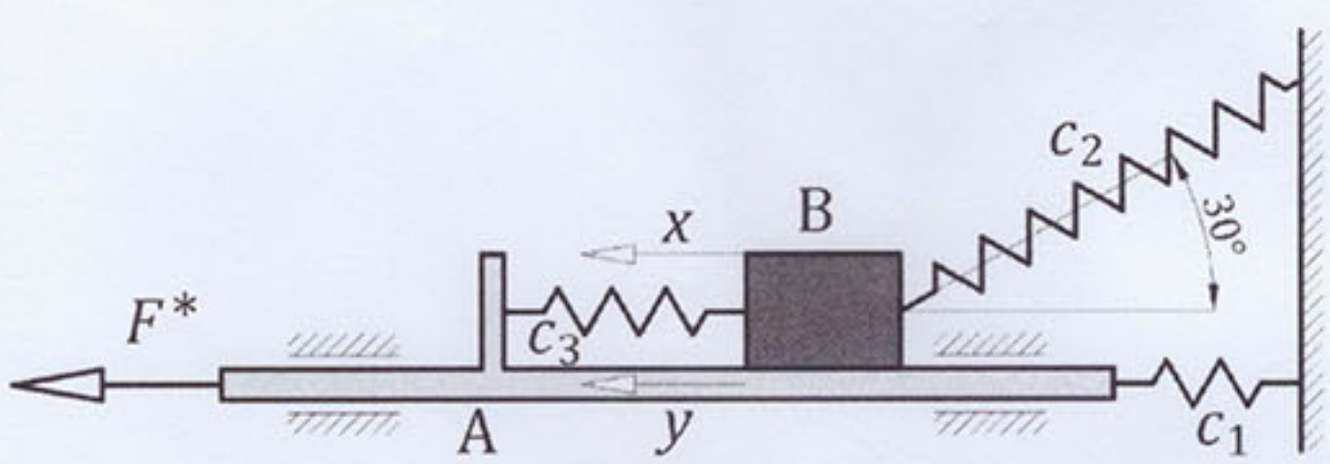
1. Осцилаторни систем, чији је равнотежни положај приказан на слици, састоји се из хомогеног штапа масе $2m$ и дужине $3l$ и лаког штапа исте дужине на чијем је крају везана концентрисана маса m. Формирати карактеристичну једначину система који врши мале осцилације у вертикалној равни у околини равнотежног положаја.

Дато је: $c_2 = 2c_1 = 50\ \text{N/m}, \beta = 40\ \text{Ns/m}, m = 4\ \text{kg}, l = 1\ \text{m}.$



2. По транслаторно покретном глатком тијелу А масе m креће се тијело В масе $2m$. Координата x описује релативни положај тијела В у односу на тијело А, док координата y дефинише положај тијела А. Ако систем врши мале осцилације у околини равнотежног положаја приказаног на слици, одредити:
   - амплитуде принудних осцилација система и
   - општу једначину осциловања.

Дато је: $c_1 = 2c_2 = 3c_3 = 90\ \text{N/m}, m = 2\ \text{kg}, F^* = 12 \sin 4t.$

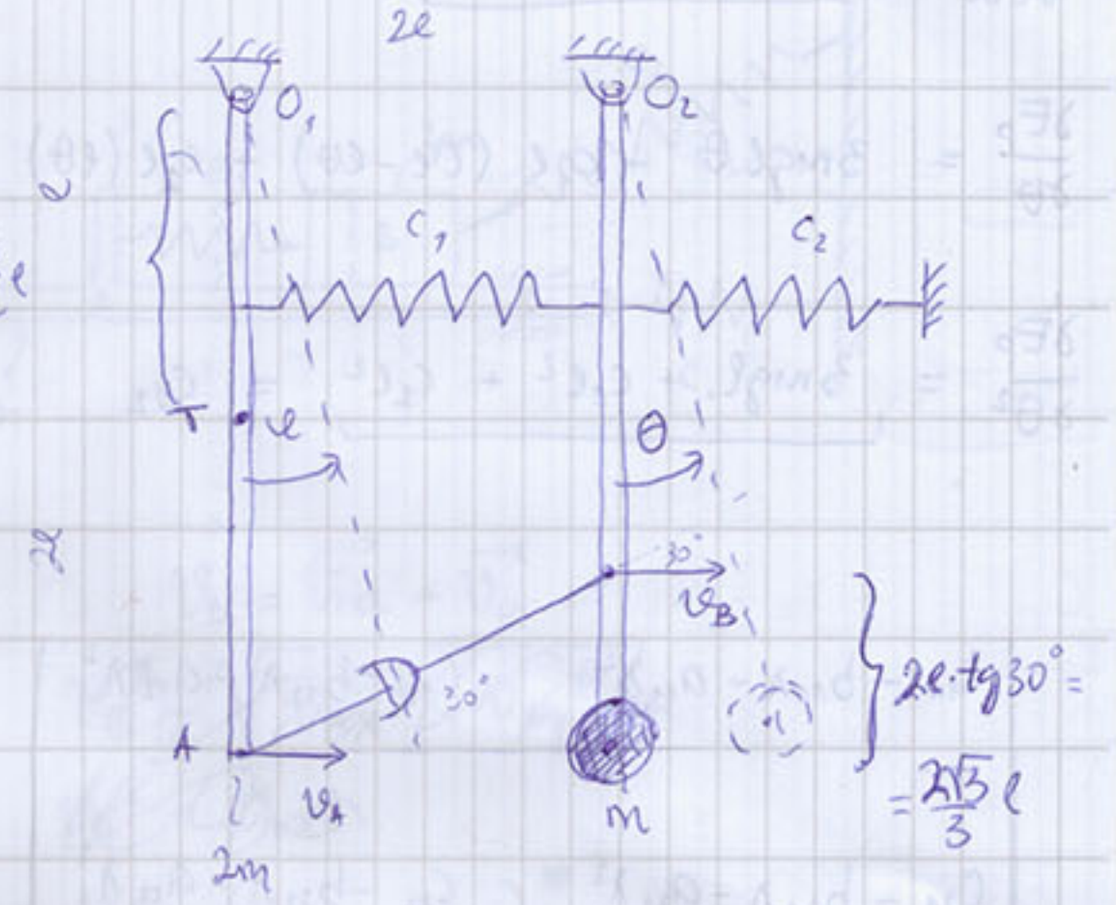


$q_1 = \varphi \qquad q_2 = \theta$

1. $m = 4\,kg \;;\; \ell = 1\,m$

$c_2 = 2c_1 = 50\,N/m$

$\beta = 40\,Ns/m$



$$E_K = \frac{1}{2} J_{O_1} \cdot \dot{\varphi}^2 + \frac{1}{2} J_{O_2} \cdot \dot{\theta}^2 =$$

$$= \frac{1}{2} \left(\frac{2m \cdot (3\ell)^2}{12} + 2m \left(\frac{3}{2}\ell \right)^2 \right) \dot{\varphi}^2 + \frac{1}{2} m (3\ell)^2 \cdot \dot{\theta}^2 =$$

$$= \frac{1}{2} \left(\frac{18}{12} m\ell^2 + \frac{9}{2} m\ell^2 \right) \dot{\varphi}^2 + \frac{1}{2} \cdot 9 m\ell^2 \dot{\theta}^2 =$$

$$= \frac{1}{2} \cdot 6 m\ell^2 \dot{\varphi}^2 + \frac{1}{2} 9 m\ell^2 \dot{\theta}^2 \quad \Rightarrow \quad \underline{a_{11} = 6m\ell^2 \;,\; a_{12} = a_{21} = 0 \;,\; a_{22} = 9m\ell^2}$$

✓ ✓ ✓

$$\Phi = \frac{1}{2} \beta v_r^2 = \frac{1}{2} \beta \left(v_A \cos 30^\circ - v_B \cos 30^\circ \right)^2 = \frac{1}{2} \beta \left(\frac{\sqrt{3}}{2} \cdot 3\ell\dot{\varphi} - \frac{\sqrt{3}}{2} \cdot \left(3\ell - \frac{2\sqrt{3}}{3}\ell \right) \dot{\theta} \right)^2 =$$

✓

$$= \frac{1}{2} \beta \left(2{,}6\,\ell\dot{\varphi} - 1{,}6\,\ell\dot{\theta} \right)^2 = \frac{1}{2} \beta \left(6{,}76\ell^2\dot{\varphi}^2 - 2 \cdot 4{,}16\ell^2\dot{\varphi}\dot{\theta} + 2{,}56\ell^2\dot{\theta}^2 \right)$$

$$\Rightarrow \underline{b_{11} = 6{,}76\ell^2\beta \;,\; b_{12} = b_{21} = -4{,}16\ell^2\beta \;,\; b_{22} = 2{,}56\ell^2\beta}$$

$$E_p = 2mg \cdot \left(\frac{3}{2}\ell - \frac{3}{2}\ell\cos\varphi \right) + mg \left(3\ell - 3\ell\cos\theta \right) + \frac{1}{2} c_1 \left(\ell\sin\varphi - \ell\sin\theta \right)^2 + \frac{1}{2} c_2 \left(\ell\sin\theta \right)^2 =$$

$$= 3mg\ell \left(1 - 1 + \frac{\varphi^2}{2} \right) + 3mg\ell \left(1 - 1 + \frac{\theta^2}{2} \right) + \frac{1}{2} c_1 \left(\ell\varphi - \ell\theta \right)^2 + \frac{1}{2} c_2 \left(\ell\theta \right)^2 =$$

$$\frac{\partial E_p}{\partial \varphi}\Big|_{\substack{\varphi=0 \\ \theta=0}} = 3mg\ell\varphi + c_1 \cdot \ell \left(\ell\varphi - \ell\theta \right)$$

$$\frac{\partial^2 E_p}{\partial \varphi^2}\Big|_{\substack{\varphi=0 \\ \theta=0}} = \underline{3mg\ell + c\ell^2 = c_{11}}$$

Код уравнения упругости:

$$\frac{\partial^2 E_p}{\partial x \partial \theta} = \underline{-c_1 \ell^2 = c_{12} = c_{21}}$$

$$a_{11} = 24, \quad a_{12} = a_{21} = 0, \quad a_{22} = 36$$

$$\frac{\partial E_p}{\partial \theta} = 3mg\ell\theta - c_1\ell(\ell x - \ell\theta) + c_2\ell(\ell\theta)$$

$$b_{11} = 270{,}4, \quad b_{12} = b_{21} = -166{,}4, \quad b_{22} = 102{,}4$$

$$\frac{\partial^2 E_p}{\partial \theta^2} = \underline{3mg\ell + c_1\ell^2 + c_2\ell^2} = c_{22}$$

$$c_{11} = 142{,}72, \quad c_{12} = c_{21} = -25, \quad c_{22} = 192{,}72$$

$$\begin{vmatrix} c_{11} - b_{11}\lambda - a_{11}\lambda^2 & c_{12} - b_{12}\lambda - a_{12}\lambda^2 \\ c_{21} - b_{21}\lambda - a_{21}\lambda^2 & c_{22} - b_{22}\lambda - a_{22}\lambda^2 \end{vmatrix} = 0$$

$$c_{11}c_{22} - c_{11}b_{22}\lambda - c_{11}a_{22}\lambda^2 - c_{22}b_{11}\lambda + b_{11}b_{22}\lambda^2 + b_{11}a_{22}\lambda^3 - c_{22}a_{11}\lambda^2 + a_{11}b_{22}\lambda^3 + a_{11}a_{22}\lambda^4 - (c_{12}c_{21} - c_{12}b_{21}\lambda - c_{12}a_{21}\lambda^2 - c_{21}b_{12}\lambda + b_{12}b_{21}\lambda^2 + b_{12}a_{21}\lambda^3 - c_{21}a_{12}\lambda^2 + a_{12}b_{21}\lambda^3 + a_{12}a_{21}\lambda^4) = 0$$

характеристика уравнения:

$$A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0 = 0$$

$$A_4 = a_{11}a_{22} = 24 \cdot 36 = 864$$

$$A_3 = b_{11}a_{22} + a_{11}b_{22} = 270{,}4 \cdot 36 + 24 \cdot 102{,}4 = 12192$$

$$A_2 = -c_{11}a_{22} + b_{11}b_{22} - c_{22}a_{11} \mp b_{12}b_{21} = -142{,}72 \cdot 36 + 270{,}4 \cdot 102{,}4 - 192{,}72 \cdot 24 - (-166{,}4)^2 = -97$$

$$A_1 = -c_{11}b_{22} - c_{22}b_{11} + c_{12}b_{21} + c_{21}b_{12} = -142{,}72 \cdot 102{,}4 - 192{,}72 \cdot 270{,}4 + (-25)\cdot(166{,}4) + (-25)(-166{,}4) = -58$$

$$A_0 = c_{11}c_{22} - c_{12}c_{21} = 142{,}72 \cdot 192{,}72 - (-25)(-25) = 26878{,}1$$

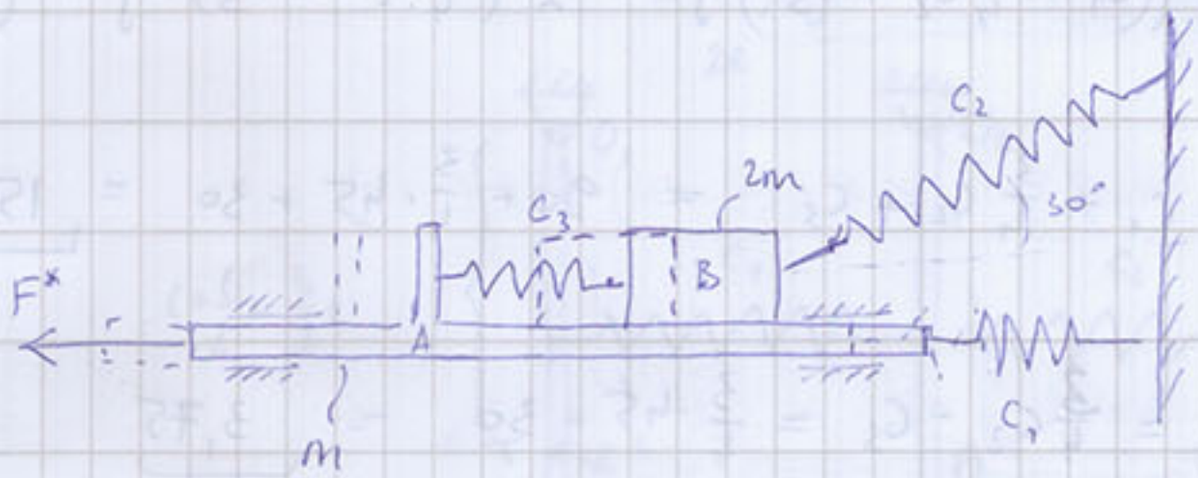
$$\underline{864\lambda^4 + 12192\lambda^3 - 9763{,}2\lambda^2 - 58406\lambda + 26878{,}1 = 0}$$

2. $q_1 = y \quad q_2 = x$

$m = 2kg$

$c_1 = 2c_2 = 3c_3 = 90 \frac{N}{m}$

$F^* = 12mNt$



$E_K = E_{KA} + E_{KB}$

$E_K^A = \frac{1}{2} m \cdot \dot{y}^2$

$E_K^B = \frac{1}{2} \cdot 2m\, v_B^2 = \frac{1}{2} \cdot 2m (\dot{x} + \dot{y})^2$ ✓

$\vec{v_B} = \vec{v_{rel}} + \vec{v_{pr}}$

$v_{B_y} = (v_{rel} + v_{pr})_y$

$v_{B_x} = (\dot{x} + \dot{y})_x$

$E_K = \frac{1}{2}\{m \dot{y}^2 + \frac{1}{2} 2m (\dot{x}^2 + 2\dot{x}\dot{y} + \dot{y}^2) = \frac{1}{2}(m + 2m)\dot{y}^2 + \frac{1}{2} 2(2m)\dot{x}\dot{y}\} + \frac{1}{2} 2m \dot{x}^2$

$\underbrace{a_{11}} = 3m = \underbrace{6}$ ✓

$\underbrace{a_{12} = a_{21}} = 2m = \underbrace{4}$

$\underbrace{a_{22}} = 2m = \underbrace{4}$

$E_p = E_p^{c_1} + E_p^{c_2} + E_p^{c_3}$

$E_p^{c_1} = \frac{1}{2} c_1 \{y^2$ ✓

$E_p^{c_2} = \frac{1}{2} c_2 ((x+y)\cos 30°)^2 =$

$= \frac{1}{2} \cdot \frac{3}{4} c_2 (x^2 + 2xy + y^2) = \frac{3}{8}$ ✓

$E_p^{c_3} = \frac{1}{2} c_3 (x - y)^2 = \frac{1}{2} c_3 (x^2 - 2xy + y^2)$

$E_p = \frac{1}{2} c_1 y^2 + \frac{3}{8} c_2 (x^2 + 2xy + y^2) + \frac{1}{2} c_3 (x^2 - 2xy + y^2) =$

$$E_p = \frac{1}{2}\left(\left(c_1 + \frac{3}{4}c_2 + c_3\right)y^2 + 2\cdot\left(\frac{3}{4}c_2 - c_3\right)xy + \left(\frac{3}{4}c_2 + c_3\right)x^2\right)$$

$$\underline{c_{11}} = c_1 + \frac{3}{4}c_2 + c_3 = 90 + \frac{3}{4}\cdot 45 + 30 = \underline{153{,}75}$$

$$\underline{c_{12} = c_{21}} = \frac{3}{4}c_2 - c_3 = \frac{3}{4}\cdot 45 - 30 = \underline{3{,}75}$$

$$\underline{c_{22}} = \frac{3}{4}c_2 + c_3 = \frac{3}{4}\cdot 45 + 30 = \underline{63{,}75}$$

$$\delta A^* = \vec{F^*}\cdot\delta\vec{r} = F^*\cdot dr\cos 0^\circ = F^*\cdot\delta y = \Omega^2 m 4t\,\delta y$$

$$\underline{Q_{o_1} = 12} \qquad \underline{Q_{o_2} = 0}$$

$$\Delta(\Omega^2) = \begin{vmatrix} c_{11} - a_{11}\Omega^2 & c_{12} - a_{12}\Omega^2 \\ c_{21} - a_{21}\Omega^2 & c_{22} - a_{22}\Omega^2 \end{vmatrix} = \begin{vmatrix} 153{,}75 - 6\cdot 4^2 & 3{,}75 - 4\cdot 4^2 \\ 3{,}75 - 4\cdot 4^2 & 63{,}75 - 4\cdot 4^2 \end{vmatrix} =$$

$$= \begin{vmatrix} 57{,}75 & -60{,}25 \\ -60{,}25 & -0{,}25 \end{vmatrix} = -57{,}75\cdot 0{,}25 - (-60{,}25)\cdot(-60{,}25) = -3631{,}33$$

$$\underline{P_1} = \frac{\begin{vmatrix} Q_{o_1} & c_{12} - a_{12}\Omega^2 \\ Q_{o_2} & c_{22} - a_{22}\Omega^2 \end{vmatrix}}{\Delta(\Omega^2)} = \frac{\begin{vmatrix} 12 & -60{,}25 \\ 0 & -0{,}25 \end{vmatrix}}{-3631{,}33} = \frac{-12\cdot 0{,}25}{-3631{,}33} = \underline{0{,}00088}$$

$$\underline{P_2} = \frac{\begin{vmatrix} 57{,}75 & 12 \\ -60{,}25 & 0 \end{vmatrix}}{\Delta(\Omega^2)} = \frac{-12(-60{,}25)}{-3631{,}33} = \underline{-0{,}12}$$

Фреквентна јед:

$$\begin{vmatrix} c_{11} - a_{11}\omega^2 & c_{12} - a_{12}\omega^2 \\ c_{21} - a_{21}\omega^2 & c_{22} - a_{22}\omega^2 \end{vmatrix} = 0$$

$$\begin{vmatrix} 153{,}75 - 6\omega^2 & 3{,}75 - 4\omega^2 \\ 3{,}75 - 4\omega^2 & 63{,}75 - 4\omega^2 \end{vmatrix} = 0$$

$$9.801{,}56 - 615\omega^2 - 382{,}5\omega^2 + 24\omega^4 - (14{,}06 - 15\omega^2 - 15\omega^2 + 16\omega^4) = 0$$

$$8\omega^4 - 967{,}5\omega^2 + 9661{,}5 = 0$$

Уводимо смену $\omega^2 = k$

$$8k^2 - 967{,}5k + 9661{,}5 = 0$$

$$k_{1,2} = \frac{967{,}5 \pm \sqrt{967{,}5^2 - 4 \cdot 9661{,}5 \cdot 8}}{16} = \frac{967{,}5 \pm 791{,}76}{16}$$

$$k_1 = 10{,}98 \qquad k_2 = 109{,}95$$

$$\underline{\omega_1} = \sqrt{k_1} = 3{,}31 \qquad \underline{\omega_2} = \sqrt{k_2} = \underline{10{,}48}$$

$$\eta_{21}^{(1)} = -\frac{c_{11} - a_{11}\omega_1^2}{c_{12} - a_{12}\omega_1^2} = -\frac{153{,}75 - 6 \cdot 10{,}98}{3{,}75 - 4 \cdot 10{,}98} = -\frac{87{,}87}{-40{,}17} = 2{,}19$$

$$\eta_{21}^{(2)} = -\frac{c_{11} - a_{11}\omega_2^2}{c_{12} - a_{12}\omega_2^2} = -\frac{153{,}75 - 6 \cdot 109{,}95}{3{,}75 - 4 \cdot 109{,}95} = -\frac{-505{,}95}{-436{,}05} = -1{,}16$$

$$y^{(1)} = A_1^{(1)} \sin(\omega_1 t + \delta_1) \qquad y^{(2)} = A_1^{(2)} \sin(\omega_2 t + \delta_2)$$

$$x^{(1)} = \eta_{21}^{(1)} A_1^{(1)} \sin(\omega_1 t + \delta_1) \qquad x^{(2)} = \eta_{21}^{(2)} A_1^{(2)} \sin(\omega_2 t + \delta_2)$$

$$y^{(1)} = A_1^{(1)} \sin(3{,}31\, t + \delta_1) \qquad y^{(2)} = A_1^{(2)} \sin(10{,}48\, t + \delta_2)$$

$$x^{(1)} = 2{,}19\, A_1^{(1)} \sin(3{,}31 t + \delta_1) \qquad x^{(2)} = -1{,}16\, A_1^{(2)} \sin(10{,}48 t + \delta_2)$$

$$y_h = y^{(1)} + y^{(2)} \quad \text{a} \quad x_h = x^{(1)} + x^{(2)}$$

$$y_p = P_1 \sin \Omega t = 0{,}0008 \sin(4t)$$

$$x_p = P_2 \sin \Omega t = -0{,}12 \sin(4t)$$

$$y = y_h + y_p \quad \text{a} \quad x = x_h + x_p$$